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LETTER TO THE EDITOR

New solutions of the Yang–Baxter equation without additivity, and its coloured interpretation

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Abstract. General forms of 4×4 solutions of the Yang-Baxter equation, without the additivity of spectral parameters, are obtained. By making a parametrization of the colours in terms of the spectral parameters we shed new light on the coloured solution of the Yang-Baxter equation.

The (4×4)-dimensional solutions of the braid relation

$$S_{12}S_{23}S_{12}=S_{23}S_{12}S_{23} \tag{1}$$

are well known. Direct calculation shows that it admits two types of solutions: standard [1-4] and exotic [5, 6]. The former is related to $\mathrm{SU}_q(2)$ and the latter to $\mathrm{SU}_q(1,1)$ [7] or the quantum algebra given in [8]. To generate the solutions of the Yang-Baxter equation (YBE)

$$\check{R}_{12}(x)\check{R}_{23}(xy)\check{R}_{12}(y)=\check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{23}(x) \tag{2}$$

where x and y are spectral parameters, two methods were proposed: either using [9]

$$\check{R}(x)=xS-x^{-1}S^{-1} \tag{3}$$

or [10]

$$\check{R}(x)^{ab}_{cd}=Q^{\alpha(d-a)u}S^{ab}_{cd} \tag{4}$$

where Q and α are arbitrary constants and $x=\exp(u)$. It is known that (4) is a special case of the 'symmetry-breaking transformation' discussed in [2, 10, 11] and this spectral-parameter 'dressing' process does not change the elements with $a=b=c=d$ (a, b, c and $d=+$ or $-$) for spin-$\frac{1}{2}$ models.

For a long time it seemed that the standard and exotic solutions

$$S_s=\begin{bmatrix} q &&& \\ & 0 & 1 & \\ & 1 & q-q^{-1} & \\ &&& q\end{bmatrix} \qquad S_e=\begin{bmatrix} q &&& \\ & 0 & 1 & \\ & 1 & q-q^{-1} & \\ &&& -q^{-1}\end{bmatrix} \tag{5}$$

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have exhausted the 4×4 solutions of (1) except that the external field can be applied to change the 1's to be η and η^{-1} [12]. However, recently Murakami [13] has found the coloured solution of (1)

$$S^{\lambda\mu}=\begin{bmatrix} t_\lambda & & & \\ & 0 & 1 & \\ & t_\lambda t_\mu & t_\mu^{-1}(t_\lambda^2+1) & \\ & & & +t_\mu^{-1} \end{bmatrix} \tag{6}$$

where λ and μ stand for the colours and t_λ, t_μ the coloured parameters. In order to obtain a solution of (2), equation (3) is used in [13]. As usual for a given solution of (1) a Yang–Baxterization prescription has been developed to generate solution of (2) [9] without colours.

Observing (6) if we regarded $S^{\lambda\mu}$ as functions of continuous variables λ and μ, namely $S^{\lambda\mu}=S(\lambda,\mu)$, then λ and μ are placed as 'spectral parameters'. We can ask a question: can we find a general form of solution of

$$\check{R}_{12}(\lambda,\mu)\check{R}_{23}(\lambda,\nu)\check{R}_{12}(\mu,\nu)=\check{R}_{23}(\mu,\nu)\check{R}_{12}(\lambda,\nu)\check{R}_{23}(\lambda,\mu) \tag{7}$$

which takes (5) and (6) as special cases? The answer is yes. In this letter we show that the concept of connection between colours and the spectral parameters in the sense of an extension of type (4) can be set up and can help to find a general form of solution of (7).

In order to solve (7) we first write the weight-conserved general form

$$\check{R}(\lambda,\mu)=\sum_{a=+,-}u_a(\lambda,\mu)E_{aa}\otimes E_{aa}+W(\lambda,\mu)E_{--}\otimes E_{++} \\ +\sum_{\substack{a\neq b\\ a,b=+,-}}p^{(a,b)}(\lambda,\mu)E_{ab}\otimes E_{ba} \tag{8}$$

where $(E_{ab})_{cd}=\delta_{ac}\delta_{bd}$, a, b, c, $d=+,\ -$ and $u_a(\lambda,\mu)$, $W(\lambda,\mu)$ and $p^{(a,b)}(\lambda,\mu)$ will be determined by (7).

Substituting (8) into (7) we obtain the relations

$$u_-(\lambda,\mu)p^{(-,+)}(\lambda,\nu)=p^{(-,+)}(\lambda,\mu)u_-(\lambda,\nu) \tag{9}$$

$$u_-(\lambda,\nu)p^{(+,-)}(\mu,\nu)=p^{(+,-)}(\lambda,\nu)u_-(\mu,\nu) \tag{10}$$

$$u_+(\lambda,\mu)p^{(+,-)}(\lambda,\nu)=p^{(+,-)}(\lambda,\mu)u_+(\lambda,\nu) \tag{11}$$

$$u_+(\lambda,\nu)p^{(-,+)}(\mu,\nu)=p^{(-,+)}(\lambda,\nu)u_+(\mu,\nu) \tag{12}$$

$$\{u_+(\lambda,\mu)u_+(\mu,\nu)-p^{(-,+)}(\lambda,\mu)p^{(+,-)}(\mu,\nu)\}\,W(\lambda,\nu)=u_+(\lambda,\nu)(\lambda,\mu)\,W(\mu,\nu) \tag{13}$$

$$(u_-(\lambda,\nu))^{-1}\{u_-(\lambda,\mu)u_-(\mu,\nu)-p^{(-,+)}(\mu,\nu)p^{(+,-)}(\lambda,\mu) \\ =(u_+|\lambda,\nu))^{-1}\{u_+(\lambda,\mu)u_+(\mu,\nu)-p^{(+,-)}(\mu,\nu)p^{(-,+)}(\lambda,\mu)\}. \tag{14}$$

Let us consider variable-separation solutions of (9)–(14). Suppose

$$\begin{aligned} u_a(\lambda,\mu)&=f_aQ^{\sum_{k=1}^{m}\{\alpha_k(a)\lambda^k+\beta_k(a)\mu^k\}} &&(a,b=+,-)\\ p^{(a,b)}(\lambda,\mu)&=h(a,b)Q^{\sum_{k=1}^{m}\{\gamma_k(a,b)\lambda^k+\sigma_k(a,b)\mu^k\}} &&(a\neq b)\quad(a=+,-) \end{aligned} \tag{15}$$

where f_a, $h(a,b)$, $\alpha_k(a)$, $\beta_k(a)$, $\gamma_k(a)$ and $\sigma_k(a,b)$ are to be determined as the consequence of (9)–(15). The parameter Q is arbitrary.

Substituting (15) into (9)–(12) leads to

$$\gamma_k(a, b) = \alpha_k(b) \qquad \sigma_k(a, b) = \beta_k(a) \qquad (a, b = +, -;\ a \neq b). \tag{16}$$

Substituting (15), (16) into (13) one obtains

$$\begin{aligned}\{f_+ Q^{\Sigma_{k=1}^{m}\{\alpha_k(+)+\beta_k(+)\}\mu^k} + f_+^{+1} Q^{\Sigma_{k=1}^{m}\{\alpha_k(-)+\beta_k(-)\}\mu^k} h(+,-)h(-,+)\}\, W(\lambda, \nu)&\\ = W(\lambda, \mu)\, W(\mu, \nu).&\end{aligned} \tag{17}$$

Then substituting them into (14) yields

$$\begin{aligned}f_- Q^{\Sigma_{k=1}^{m}\{\alpha_k(-)+\beta_k(-)\}\mu^k} + f_-^{-1} h(+,-)h(-,+) Q^{\Sigma_{k=1}^{m}\{\alpha_k(+)+\beta_k(+)\}\mu^k}&\\ = f_+ Q^{\Sigma_{k=1}^{m}\{\alpha_k(+)+\beta_k(+)\}\mu^k} - f_+^{-1} h(+,-)h(-,+) Q^{\Sigma_{k=1}^{m}\{\alpha_k(-)+\beta_k(-)\}\mu^k}&\end{aligned} \tag{18}$$

which allows two types of solutions:

$$\text{(i)} \qquad f_+ = f_- \qquad \alpha_k(+) - \alpha_k(-) = \beta_k(-) - \beta_k(+) \tag{19}$$

$$\text{(ii)} \qquad f_+ f_- = +h(+,-)h(-,+). \tag{20}$$

Since the YBE allows an overall factor one can take

$$h(+,-) = \eta \qquad h(-,+) = \eta^{-1}. \tag{21}$$

In terms of the matrix form, equation (8) leads to

$$\check{R}(\lambda, \mu) = \begin{bmatrix} u_+(\lambda, \mu) & & & \\ & 0 & p^{(+,-)}(\lambda, \mu) & \\ & p^{(-,+)}(\lambda, \mu) & W(\lambda, \mu) & \\ & & & u_-(\lambda, \mu) \end{bmatrix} \tag{22}$$

where the parameters in (22) obey the relations (15)–(18). Obviously (15) and (16) give

$$u_a(\lambda, \mu) = f_a \varphi_a(\lambda, \mu) \qquad (a, b = +, -) \tag{23}$$

$$p^{(a,b)}(\lambda, \mu) = h(a, b)\varphi_a(\lambda, \mu) \qquad (a \neq b) \tag{24}$$

where

$$\varphi_a(\lambda, \mu) = Q^{\Sigma_{k=1}^{m}\{\alpha_k(a)\lambda^k + \beta_k(a)\mu^k\}} \tag{25}$$

and the $W(\lambda, \mu)$ satisfy (17).

Let us now discuss the type (i) solutions, denoting by $\check{R}_I(\lambda, \mu)$. Taking $f_+ = f_- = q$ and noting that $W(\lambda, \mu)$ satisfy

$$W(\lambda, \mu)\, W(\mu, \nu) = \varphi_+(\lambda, \mu)(q - q^{-1})\, W(\lambda, \nu) \tag{26}$$

we derive

$$\check{R}_I(\lambda, \mu) = \varphi_+(\lambda, \mu) \begin{bmatrix} q & & & \\ & 0 & \eta X & \\ & Y^{-1}\eta^{-1} & \bar{W}(\lambda, \mu) & \\ & & & qXY^{-1} \end{bmatrix} \tag{27}$$

where

$$\begin{aligned} &X = Q^{\Sigma_{k=1}^{m}\bar{\alpha}_k\lambda^k} \qquad && Y = Q^{\Sigma_{k=1}^{m}\bar{\alpha}_k\mu^k} \\ &\bar{\alpha}_k = \alpha_k(-) - \alpha_k(+) \qquad && \bar{\beta}_k = \beta_k(-) + \beta_k(+) \end{aligned} \tag{28}$$

and

$$W(\lambda, \mu) = \varphi_-(\lambda, \mu)\,\bar{W}(\lambda, \mu) \tag{29}$$

$$\bar{W}(\lambda, \mu)\,\bar{W}(\mu, \nu) = (q - q^{-1})\,\bar{W}(\lambda, \mu). \tag{30}$$

A simple choice is

$$\bar{W}(\lambda, \mu) = (q - q^{-1})(g(\lambda)/g(\mu)) \tag{31}$$

with $g(\lambda)$ and $g(\mu)$ being arbitrary functions of λ and μ. We emphasize that by virtue of (30) equation (27) provides a non-additivity solution of (7) where the spectral-parameter dressing process for a solution of (1) is an extension of type (4) rather than type (3). In other words the spectral parameters in (27) appear in separate elements and are placed as the colours. Under such an understanding (27) can be further Yang–Baxterized by using (3) to generate a coloured x-solution of (2). It seems that for a solution of the braid relation we can Yang–Baxterize it twice. The first is by direct computation (8) and without additivity, like (7). The second process is in the usual sense by using (3) with the additivity obeying (2). In order to understand the statement let us consider some particular cases.

When $\lambda = \mu = 0$, equation (27) is reduced to

$$\check{R}_I(0, 0) = \begin{bmatrix} q & & & \\ & 0 & \eta & \\ & \eta^{-1} & q - q^{-1} & \\ & & & q \end{bmatrix} \tag{32}$$

where $u_+(0, 0) = u_-(0, 0) = q$, $p^{(+,-)}(0, 0) = \eta$, $p^{(-,+)}(0, 0) = \eta^{-1}$, and $W(0, 0) = q - q^{-1}$. Equation (32) is exactly the standard solution with an external field. It has been discussed in [12].

When $W(\lambda, \mu)$ are dependent on the difference of λ and μ and $x = Q^{\lambda-\mu}$ then

$$W(\lambda, \mu) = (q - q^{-1})x \tag{33}$$

and we get

$$\check{R}_I(x) = \varphi_+(\lambda, \mu) \begin{bmatrix} q & & & \\ & 0 & \eta & \\ & \eta^{-1} & (q - q^{-1})x & \\ & & & q \end{bmatrix} \tag{34}$$

which satisfies (2) through (4).

Taking $f_+ = q$ and noting that (20) gives rise to $f_- = -q^{-1}$ we get the type (ii) solution of (7)

$$\check{R}_{II}(\lambda, \mu) = \begin{bmatrix} q & & & \\ & 0 & X\eta & \\ & \eta^{-1}Y^{-1} & \bar{W}(\lambda, \mu) & \\ & & & -q^{-1}XY^{-1} \end{bmatrix} \tag{35}$$

where X and Y have are given by (28).

When (33) holds and $\bar{\alpha}_k = \bar{\beta}_k = 0$ one gets

$$\check{R}_{II}(x) = \begin{bmatrix} q & & & \\ & 0 & \eta & \\ & \eta^{-1} & (q-q^{-1})x & \\ & & & -q^{-1} \end{bmatrix}. \tag{36}$$

This is the exotic solution of the braid relation plus an external field and equation (4).

When $\alpha_k(-) = \beta_k(+) = 0$ and $\alpha_k(+) = -\beta_k(-)$ we obtain (6) with

$$t_\lambda = qQ^{\sum_{k=1}^{m} \alpha_k(+)\lambda^k} \qquad t_\mu = qQ^{\sum_{k=1}^{m} \alpha_k(-)\mu^k} \qquad \eta = 1 \tag{37}$$

$$\bar{W}(\lambda, \mu) = t_\lambda^{-1} t_\mu^{-1}(t_\lambda^2 - 1). \tag{38}$$

Therefore Murakami's solution [13] is a particular case of the general form of solutions (35).

We see that there exist two types of coloured solutions shown by (27) and (35).

Through this simple example we conclude that if we use the extended version of type (4) to generate (λ, μ)-dependent solutions of the YBE then it can be non-additive. After this doing if we regard the spectral parameters (λ, μ) with non-additivity for YBE as the colours then one can 'secondly' Yang–Baxterize the coloured solutions to obey (2).

It is surprising that in such a simple example there appears the 'non-additive' solutions of the YBE, since we know that complicated calculations were made for deriving the non-additivity of spectral parameters for the YBE in, for example, [14–16]. This may be because we are looking for simpler solutions with non-additive type of spectral parameters operated by direct calculation through (8).

We would like to point out that in general the derived general solutions do not satisfy the usual unitarity conditions for any λ and μ even though some of the particular cases do. This is because the general solutions are beyond the scope at (4). This problem and more complicated examples will be discussed in a following paper.

Finally we emphasize that either the standard solution (27) or the exotic solution (35) can be derived fom the quantum double of Drinfeld construction where $X(\lambda)$ and $Y(\mu)$ come from the allowed Cartan centralizer.

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